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An L^2-theory on SPDE driven by Lévy processes

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Abstract

In this paper we develop an L_2-theory for stochastic partial differential equations driven by Lévy processes. The coefficients of the equations are random functions depending on time and space variables, and no smoothness assumption of the coefficients is assumed.

Keywords: Stochastic parabolic partial differential equations, Lévy processes.

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1 Introduction

In this article we study the L^2-theory of stochastic partial differential equations of the following type:

$$du = \left(\frac{\partial}{\partial x_i} \left(a^{ij} u_{x^j} + \bar{b}^i u \right) + b^i u_{x^i} + cu + f \right) dt + \left(\sigma^{ik} u_{x^i} + \mu^k u + g^k \right) dZ_t^k \tag{1.1}$$

given for $t \geq 0$ and $x \in \mathbb{R}^d$. Here $\{Z_t^k,\ k = 2, 1, \cdots\}$ are independent one-dimensional Lévy processes, i and j go from 1 to d with the summation convention on i, j, k being enforced. For example, the second term in the right hand side of (1.1) should be understood as

$$\sum_{k \geq 1} \left(\sum_{i=1}^{d} a^{ik} u_{x^i} + \mu^k u + g^k \right) dZ_t^k.$$

The coefficients a^{ij}, b^i, c, σ^{ik}, μ^k and the free terms f, g^k are **random** functions depending on (t, x).

Stochastic partial differential equations (SPDEs) of type (1.1) arise naturally in applications when the objects are subject to randomness and high variability. The purpose of this paper is to investigate the existence and uniqueness of pathwise solutions to (1.1) and to study the regularity of the solutions.

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If Z_t^k are independent one-dimensional Wiener processes, then general L^p-theory of the equation has been well studied. An L^p-theory of SPDEs with Wiener processes defined on $\mathbb{R}^n$ was first introduced by Krylov in [7], and in [9] and [10] Krylov and Lototsky developed an L^p-theory of such equations with constant coefficients defined on half space $\mathbb{R}^n_+$. Later in many articles (see [6], [5] and references therein) these results were extended for SPDEs with variable coefficients defined on bounded domains of $\mathbb{R}^n$.

However very little is known when Z_t^k are general discontinuous Lévy processes. In [2], existence and uniqueness of weak (or martingale) solutions as well as pathwise solutions to the following SPDE

$$du = \mathcal{A}udt + \sum_{k=1}^{n} g^k(u)dZ_t^k, \tag{1.2}$$

driven by Lévy processes is studied, where $\mathcal{A}$ is the generator of certain semigroup on a Hilbert space H and g^k, $k = 1, \cdots, n$, are non-random maps from H to H that satisfy certain continuity condition.

See the Introduction of [2] for a brief discussion on other related work SPDE driven by Poisson random measure or stable noises, including [1, 3, 11, 12]. Note that maps g^k, $k = 1, \cdots, n$, in (1.2) are **non-random** coefficients and are **independent** of t, while g^k's in (1.1) to be considered in this paper are random and time dependent but are given a priori that do not depend on solution u. Moreover no derivatives of the solution u appear in the stochastic part of equation (1.2).

Our main result, Theorem 2.11, is presented and proved in section 2. Here we show that if each Z_t^k has finite second moment, i.e., if

$$\int_{\mathbb{R}} z^2 \nu_k(dz) < \infty \quad \text{for every } k \geq 1, \tag{1.3}$$

where ν^k is the Lévy measure of Z^k, then equation (1.1) admits a unique solution in $\mathbb{H}^1(T) := L^2(\Omega \times [0,T], W_2^1)$ and the $\mathbb{H}^1(T)$-norm of the solution is controlled by the L^2-norm of f and g. In section 3 we give two extensions of Theorem 2.11. First we develop an L^2-theory for a certain type of nonlinear equations. Second, we weaken condition (1.3) by assuming that it holds only for sufficiently large k (thus it can be dropped if only finitely many processes Z^k appear in the equation) and prove that the equation has unique pathwise W_2^1-valued solution.

As usual, throughout this paper, $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$. For $i = 1, ..., d$, multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \{0, 1, 2, ...\}$, and functions $u(x)$, we set

$$u_{x^i} = \partial u / \partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + ... + \alpha_d.$$

We also use the notation D^m for a partial derivative of order m with respect to x. If we write $c = c(...)$, it means that the constant c depends only on what are in parenthesis.

2 Main results

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ satisfying the usual condition. We assume that on Ω we are given independent one-dimensional Levy processes $Z_t^1, Z_t^2, ...$ relative to $\{\mathcal{F}_t, t \geq 0\}$. Let $\mathcal{P}$ be the predictable σ-field generated by $\{\mathcal{F}_t, t \geq 0\}$.

For $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, define

$$N_k(t, A) = \left\{0 \leq s \leq t;\, Z_s^k - Z_{s-}^k \in A\right\}, \quad \widetilde{N}_k(t, A) = N_k(t, A) - t\nu_k(A)$$

where $\nu_k(A) := \mathbb{E}[N_k(1, A)]$ is the Lévy measure of Z^k. By Lévy-Itô decomposition, there exist constants α^k, β^k and Brownian motion B^k so that

$$Z_t^k = \alpha^k t + \beta^k B_t^k + \int_{|z|<1} z\widetilde{N}_k(t, dz) + \int_{|z|\geq 1} zN_k(t, dz). \tag{2.1}$$

Assumption 2.1 (i) *For each $k \geq 1$,*

$$\widehat{c}_k := \left[\int_{\mathbb{R}} z^2 \nu_k(dz)\right]^{1/2} < \infty. \tag{2.2}$$

(ii) *There exist constants $\delta, K > 0$ so that for every $t > 0$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$,*

$$\delta|\xi|^2 \leq (a^{ij} - \alpha^{ij})\xi^i\xi^j \leq a^{ij}\xi^i\xi^j \leq K|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \tag{2.3}$$

where $\alpha^{ij} := \frac{1}{2}(\widehat{c}_k^2 + \beta_k^2)\sigma^{ik}\sigma^{jk}$. Here i and j go from 1 to d, and k runs through $\{1, 2, \cdots\}$.

Recall that throughout the article, summation convention is used. Due to (2.2), $\int_{|z|>1} |z|N_k(1, dz) < \infty$, and thus by absorbing $\widetilde{\alpha}_k := \int_{|z|>1} zN_k(1, dz)$ into α_k we can rewrite (2.1) as

$$Z_t^k = \tilde{\alpha}_k t + \beta_k B_t^k + \int_{\mathbb{R}^1} z\widetilde{N}_k(t, dz).$$

For $d \geq 1$, consider the equation for random function $u(t, x)$ on $\Omega \times [0, \infty) \times \mathbb{R}^d$:

$$du = \left(\frac{\partial}{\partial x_i}\left(a^{ij}u_{x^j} + \bar{b}^i u\right) + b^i u_{x^i} + cu + f\right) dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k\right) dZ_t^k \tag{2.4}$$

in the weak sense. See Definition 2.4 below. The coefficients $a^{ij}, \bar{b}^i$, b^i, c, σ^{ik}, μ^k and the free terms f, g^k are random functions depending on $t > 0$ and $x \in \mathbb{R}^d$. Without loss of generality, we assume that $\tilde{\alpha}^k = 0$, since otherwise we can simply move the term $\sum_k \tilde{\alpha}_k \left(\sigma^{ik}u_{x^i} + \nu^k u + g^k\right) dt$ from the stochastic part to the deterministic one.

Remark 2.2 Conditions (2.2) and (2.3) will be weakened in section 3. In particular, one can completely drop the condition (2.2) if there are only finitely many processes Z_t^k in equation (2.4).

For $n = 0, 1, 2, ...,$ let

$$H^n := \left\{ u \in L^2(\mathbb{R}^d) : \, Du, ..., D^n u \in L^2(\mathbb{R}^d) \right\},$$

which is equipped with norm $\|u\|_{H^n} := \left(\sum_{k=0}^n \|D^k u\|^2_{L^2(\mathbb{R}^d)} \right)^{1/2}$. Here $Du := (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_d})$ denotes the gradient of u in the distributional sense, $D^2 u$ denotes the collection of all second derivatives of u in the distribution sense, and so on. Let $H^{-n} := (H^n)^*$ be its topological dual, and $\mathcal{P}^{dP \times dt}$ be the completion of $\mathcal{P}$ with respect to $dP \times dt$. For $n \in \mathbb{Z}$ and $T > 0$, we write $u \in \mathbb{H}^n(T)$ if u is an H^n-valued $\mathcal{P}^{dP \times dt}$-measurable process defined on $\Omega \times [0, T]$ so that

$$\|u\|_{\mathbb{H}^n(T)} := \left(\mathbb{E} \left[\int_0^T \|u(t, \cdot)\|^2_{H^n} \, dt \right] \right)^{1/2} < \infty.$$

Denote $\mathbb{L}(T) := \mathbb{H}^0(T)$. For an ℓ^2-valued processes $g = (g_1, g_2, ...)$, we say $g \in \mathbb{L}(T, \ell^2)$ if $g^k \in \mathbb{L}(T)$ for every $k \geq 1$ and

$$\|g\|_{\mathbb{L}(T,\ell^2)} := \sum_{k=1}^{\infty} (\beta_k^2 + \widehat{c}_k^2) \left(\mathbb{E} \left[\int_0^T \|g^k\|^2_{L^2} \, dt \right] \right)^{1/2} < \infty.$$

Finally we use U_2 to denote the family of $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_0$-measurable random variables u_0 having

$$\|u_0\|_{U_2} := \left(\mathbb{E} \left[\|u_0\|^2_{L^2} \right] \right)^{1/2} < \infty.$$

Remark 2.3 (i) Since we assume $\tilde{\alpha}_k = 0$, Z^k is a square integrable martingale, whose quadratic variational process will be denoted as $[Z^k]$. By Lévy system, the predictable dual projection $\langle Z^k \rangle$ of $[Z^k]$ is given by $\langle Z^k \rangle_t = (\widehat{c}_k^2 + \beta_k^2) \, t$. For every process H in $L^2(\Omega \times [0, T])$, which has a predictable $dP \times dt$-version $\widetilde{H}$, $M_t = \int_0^t H_s dZ^k_s := \int_0^t \widetilde{H}_s dZ^k_s$ is well defined and is independent of the choice of such $\widetilde{H}$, and M is a martingale with

$$\mathbb{E} \left[M_t^2 \right] = \mathbb{E} \left[\int_0^t H_s^2 \, d[Z^k]_s \right] = (\beta_k^2 + \widehat{c}_k^2) \, \mathbb{E} \left[\int_0^t H_s^2 ds \right], \quad t \leq T.$$

We will simply denote M by $\int_0^\cdot H_s dZ^k_s$. For $g = (g_1, g_2, \cdots) \in \mathbb{L}(T, \ell_2)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\sum_{k=1}^{\infty} \int_0^T (\beta_k^2 + \widehat{c}_k^2)(g^k, \phi)^2 ds \leq \|\phi\|^2_{L^2} \|g\|^2_{\mathbb{L}(T,\ell^2)} < \infty \quad \text{a.s.}$$

Therefore the series of stochastic integral $\sum_{k=1}^{\infty} \int_0^t (g^k, \phi) \, dZ^k_t$ converges uniformly in $t \in [0, T]$ in probability.

(ii) In many other articles, the equation of the type

$$du = (Au + f)dt + g(u(t-))dZ_t$$

has been studied. The expression $u(t-)$ is used so that it is predictable and the integral $\int_0^t g(u(t-))dZ_t$ becomes a martingale. Such notation is not used in (2.4), because by (i) and (ii), stochastic integral can be defined for a process H in $L_2(\Omega\times[0,T])$ which has a predictable version $\tilde{H}$, and

$$\int_0^t H(s)dZ_s = \int_0^t \tilde{H}(s)dZ_s.$$

Definition 2.4 *We say $u \in \mathcal{H}^1(T)$ if $u \in \mathbb{H}^1(T)$, u is right continuous having left limits in L^2 a.s. with $u(0) \in U_2$, and for some $f \in \mathbb{H}^{-1}(T)$ and $g = (g_1, g_2, \cdots) \in \mathbb{L}(T, \ell^2)$*

$$du(t) = f(t)dt + g^k(t)dZ_t^k \qquad \text{for } 0 \le t \le T$$

in the sense of distributions; that is, for any $\phi \in C_c^\infty(\mathbb{R}^d)$, the equality

$$(u(t), \phi) = (u(0), \phi) + \int_0^t (f(s), \phi)ds + \sum_{k=1}^\infty \int_0^t (g^k(s), \phi)dZ_s^k \tag{2.5}$$

holds for all $t \le T$ a.s.. In this case, we write

$$\mathbb{D}u := f, \quad \mathbb{S}u := g,$$

and define

$$\|u\|_{\mathcal{H}^1(T)} := \|u\|_{\mathbb{H}^1(T)} + \|\mathbb{D}u\|_{\mathbb{H}^{-1}(T)} + \|\mathbb{S}u\|_{\mathbb{L}(T,\ell^2)} + \|u(0)\|_{U_2}.$$

Lemma 2.5 *Let $u \in \mathcal{H}^1(T)$, then*

(i) *for any $\phi \in H^1$, $(u(t), \phi)$ is progressively measurable, right continuous having left limits ;*

(ii) *for each fixed $t > 0$, $u(t) = u(t-)$ in L^2 a.s.*

Proof. (i) follows immediately from (2.5).

(ii). By assumption $u(t-)$ exists. Let $\{\phi_n, : \phi_n \in H^1, n = 1, 2, ...\}$ be a orthonormal basis in $L^2(\mathbb{R}^d)$. Then the process $t \mapsto (u(t-), \phi_n)$ is predictable by (i). Since $\int_0^t (g^k, \phi_n)dZ_t^k$ is stochastically continuous, we have for each fixed t and $n \ge 1$, $(u(t), \phi_n) = (u(t-), \phi_n)$ a.s. Therefore

$$u(t-) = \sum_n (u(t-), \phi_n)\phi_n = u(t) \quad \text{a.s.}$$

The lemma is now proved. □

Theorem 2.6 *The space $\mathcal{H}^1(T)$ is a Banach space and*

$$\mathbb{E}\left[\sup_{t\le T} \|u(t)\|_{L^2}^2\right] \le c\left(\|Du\|_{\mathbb{L}(T)}^2 + \|\mathbb{D}u\|_{\mathbb{H}^{-1}(T)}^2 + \|\mathbb{S}u\|_{\mathbb{L}(T,\ell^2)}^2 + \mathbb{E}\|u(0)\|_{L^2}^2\right), \tag{2.6}$$

where c is independent of u and T.

Proof. First we prove (2.6). Let $u(0)=u_0$ and $du = f dt + g^k dZ^k_t$. Then for any $\phi \in C_0^\infty$,

$$(u(t),\phi) = (u(0),\phi) + \int_0^t (f(s),\phi)ds + \int_0^t (g^k(s),\phi)dZ^k_t \tag{2.7}$$

for all $t \le T$ (a.s.). For $f \in \mathbb{H}^{-1}(T)$, we can write it as

$$f = f_0 + \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i$$

with $f_i \in \mathbb{L}(T)$ for $0 \le i \le d$ and

$$\sum_{i=0}^d \|f_i\|_{\mathbb{L}(T)} \le c\|f\|_{\mathbb{H}^{-1}(T)}.$$

Indeed, since $f = (1-\Delta)(1-\Delta)^{-1}f$ and $(1-\Delta)^{-1} : H^n \to H^{n+2}$ is an isometry, we can take

$$f_0 = (1-\Delta)^{-1}f \qquad \text{and} \qquad f_i = -\frac{\partial f_0}{\partial x^i} \quad \text{for} \;\; i = 1,2,...,d.$$

Take a nonnegative function $\psi \in C_0^\infty(B_1(0))$ with unit integral, and for $\varepsilon > 0$ define $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$. For any generalized function u, define $u^{(\varepsilon)}(x) = u * \psi_\varepsilon(x) := (u(\cdot),\psi_\varepsilon(x-\cdot))$, then $u^{(\varepsilon)}(x)$ is infinitely differentiable function of x. By plugging $\psi_\varepsilon(x-\cdot)$ instead of ϕ in (2.7),

$$u^{(\varepsilon)}(t,x) = u^{(\varepsilon)}(0,x) + \int_0^t (f_0^{(\varepsilon)} + D_i f_i^{(\varepsilon)})dt + \int_0^t g^{(\varepsilon)k} dZ^k_t.$$

By taking $\varepsilon \to 0$, one can easily show that (2.6) holds true if for any $\varepsilon > 0$ it holds with $u^{(\varepsilon)}, u_0^{(\varepsilon)}, f^{(\varepsilon)}, g^{(\varepsilon)}$ in place of u, u_0, f, g, respectively. Thus we may assume that u, f, g are infinitely differentiable in x, and therefore (a.s.)

$$u(t) = u_0 + \int_0^t f dt + \int_0^t g^k dZ^k_t, \quad \forall t \le T. \tag{2.8}$$

The stochastic integral in (2.8) doesn't change if we replace g by its predictable version, thus we also assume that g is predictable.

Applying Ito's formula to $|u(t)|^2$ (cf. [4]) and integrating over $\mathbb{R}^d$, we have

$$\begin{aligned}
\|u(t)\|^2_{L^2} &= \|u_0\|_2^2 + 2\int_0^t (u(s),f(s))ds + \sum_k \beta_k^2 \int_0^t |g^k(s)|^2_{L^2} ds \\
&\quad +2\sum_k \int_0^t (u(s-),g^k(s))dZ^k_s + \sum_k \sum_{0<s\le t} \|g^k(s)\Delta Z^k_s\|^2_{L^2} \\
&= \|u_0\|^2_{L^2} + 2\int_0^t \left((u(s),\ f_0(s)) - \sum_{i=1}^d (u_{x^i}(s), f_i(s)) \right) ds + \sum_k \beta_k^2 \int_0^t |g^k(s)|^2_{L^2} ds \\
&\quad +2\sum_k \int_0^t (u(s-),g^k(s))dZ^k_s + \sum_k \sum_{0<s\le t} \|g^k(s)\Delta Z^k_s\|^2_{L^2},
\end{aligned} \tag{2.9}$$

where we have used the fact that Z^k's are independent and so with probability one at most one of the $Z_s^1, Z_s^2 \cdots$ can jump at any given time. By virtue of the Lévy system of the Lévy process Z_s^k, it follows that

$$\sum_{0<s\le t} \|g^k(s)\Delta Z_s^k\|_{L^2}^2 = M_t^k + \widehat{c}_k^2 \int_0^t \|g^k\|_{L^2}^2 ds, \tag{2.10}$$

where M^k is a purely discontinuous square integrable martingale with

$$M_t^k - M_{t-}^k = \|g^k(t)\Delta Z_t^k\|_{L^2}^2 \qquad \text{for } t>0.$$

It is easy to see that for every $\varepsilon>0$, there is a constant $c(\varepsilon)>0$, independent of u and f_i's such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}\left|\int_0^t \Big((u(s),f_0(s)) - \sum_{i=1}^d (u_{x^i}(s), f_i(x))\Big)\, ds\right|\right] \\
\le\;& \varepsilon\|Du\|_{\mathbb{L}(T)}^2 + \varepsilon\mathbb{E}\sup_{t\le T}\|u(t)\|_{L^2}^2 + c(\varepsilon)\sum_{i=0}^d \|f^i\|_{\mathbb{L}(T)}^2 \\
\le\;& \varepsilon\|Du\|_{\mathbb{L}(T)}^2 + \varepsilon\mathbb{E}\sup_{t\le T}\|u(t)\|_{L^2}^2 + c(\varepsilon)\|f\|_{\mathbb{H}^{-1}(T)}^2.
\end{aligned}$$

By Davis (first) inequality and Lévy system,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t}|M_s^k|\right] &\le 2\sqrt{6}\,\mathbb{E}\left[[M^k,M^k]_t^{1/2}\right] \le 2\sqrt{6}\,\mathbb{E}\left[\sum_{0<s\le t}\|g^k(t)\Delta Z_t^k\|_{L^2}^2\right] \\
&\le 2\sqrt{6}\,\widehat{c}_k^2\,\mathbb{E}\left[\int_0^t \|g^k(s)\|_{L^2}^2 ds\right]
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\sum_{k=1}^\infty\left|\int_0^s (u(r-),g^k(r))dZ_r^k\right|\right] \\
\le\;& 2\sqrt{6}\sum_{k=1}^\infty \mathbb{E}\left[\left(\sum_{0<s\le t}(u(s-),g^k(s))^2(\Delta Z_s^k)^2\right)^{1/2}\right] \\
\le\;& 2\sqrt{6}\sum_{k=1}^\infty \mathbb{E}\left[\sup_{s\le t}\|u(s)\|_{L^2}\left(\sum_{0<s\le t}\|g^k(s)\|_{L^2}^2(\Delta Z_s^k)^2\right)^{1/2}\right] \\
\le\;& \varepsilon\mathbb{E}\left[\sup_{s\le t}\|u(s)\|_{L^2}^2\right] + c(\varepsilon)\sum_{k=1}^\infty \mathbb{E}\left[\sum_{0<s\le t}\|g^k(s)\|_{L^2}^2(\Delta Z_s^k)^2\right] \\
\le\;& \varepsilon\mathbb{E}\left[\sup_{s\le t}\|u(s)\|_{L^2}^2\right] + c(\varepsilon)\sum_{k=1}^\infty \widehat{c}_k^2\,\mathbb{E}\int_0^t \|g^k(s)\|_{L^2}^2 ds.
\end{aligned}$$

It follows from (2.9) that

$$\mathbb{E}\left[\sup_{t\le T}\|u(t)\|_{L^2}^2\right] \quad\le\quad \varepsilon\mathbb{E}\left[\sup_{s\le T}\|u(s)\|_{L^2}^2\right]+\mathbb{E}\|u_0\|_{L^2}^2+\varepsilon\|Du\|_{\mathbb{L}(T)}^2+c(\varepsilon)\|f\|_{\mathbb{H}^{-1}(T)}^2+c(\varepsilon)\|g\|_{\mathbb{L}(T,\ell^2)}^2.$$

Thus (2.6) is proved if one chooses $\varepsilon\le 1/2$. Now we prove the completeness of the space $\mathcal{H}^1(T)$. Let $\{u_n : n=1,2,...\}$ be a Cauchy sequence in $\mathcal{H}^1(T)$. Let $f_n := \mathbb{D}u_n$, $g_n := \mathbb{S}u_n$ and $u_{n0} := u_n(0)$. Then there exist $u\in\mathbb{H}^1(T)$, $f\in\mathbb{H}^{-1}(T)$, $g\in\mathbb{L}(T,\ell^2)$ and $u_0\in U^2$ so that u_n, f_n, $g_n=\{g_n^k, k\ge 1\}$ and u_{n0} converge to u, f, g and u_0, respectively. Let $\phi\in C_c^\infty$ be fixed. Since

$$(u_n(t),\phi)=(u_{n0},\phi)+\int_0^t (f_n(s),\phi)ds+\sum_{k\ge1}\int_0^t (g_n^k(s),\phi)dZ_s^k,$$

taking $n\to\infty$, we have for each $t>0$,

$$(u(t),\phi)=(u_0,\phi)+\int_0^t (f(s),\phi)ds+\sum_{k\ge1}\int_0^t (g^k(s),\phi)dZ_s^k \quad \text{a.s.} \tag{2.12}$$

Since we already proved

$$\mathbb{E}\left[\sup_{t\le T}\|u_n-u_m\|_{L^2}^2\right]\le c\|u_n-u_m\|_{\mathcal{H}^1(T)}^2,$$

we conclude that $(u_n(t),\phi)$ is uniformly Cauchy in $t\in[0,T]$, (2.12) holds for all $t\le T$ a.s., and u is right continuous having left limits in L^2 a.s. Consequently $u\in\mathcal{H}^1(T)$. □

Assumption 2.7 (i) *The coefficients $a^{ij}, \bar{b}^i, b^i, c, \sigma^{ik}$ and μ^k are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions.*

(ii) *For each ω, t, x, i, j,*

$$|a^{ij}|+|\bar{b}^i|+|b^i|+|c|+\Big(\sum_{k=1}^{\infty}(\beta_k^2+\hat{c}_k^2)(|\sigma^{ik}|^2+|\mu^k|^2)\Big)^{1/2}\le K.$$

Lemma 2.8 *(A priori estimate) Let Assumptions 2.1 and 2.7 hold. Then for every solution $u\in\mathcal{H}^1(T)$ of equation* (2.4)*, we have*

$$\|u\|_{\mathcal{H}^1(T)}\le ce^{cT}\left(\|f\|_{\mathbb{H}^{-1}(T)}+\|g\|_{\mathbb{L}(T,\ell^2)}+\|u_0\|_{U_2}\right), \tag{2.13}$$

where $c=c(\delta,K)$.

Proof. We proceed as in the proof of Theorem 2.6. As before, rewrite $f\in\mathbb{H}^{-1}(T)$ as

$$f=f_0+\sum_{i=1}^d\frac{\partial}{\partial x_i}f_i \quad \text{with} \quad f^i\in\mathbb{L}(T)$$

and

$$\sum_{i=0}^{d}\|f_i\|_{\mathbb{L}(T)}\le c\|f\|_{\mathbb{H}^{-1}(T)}.$$

As in the proof of Theorem 2.6, without loss of generality, we may and do assume that u,f,g are sufficiently smooth in x. By h^k we denote the predictable version of $\sigma^{ik}u_{x^i}+\nu^k u+g^k$. By Ito's formula (cf. [4]), we have

$$\begin{aligned}
\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] &= \mathbb{E}\left[\|u_0\|_{L^2}^2\right]+2\mathbb{E}\left[\int_0^t\left(-(a^{ij}u_{x^j}+\bar{b}^iu+f_i,u_{x^i})_{L^2}+(b^iu_{x^i}+cu+f_0,u)_{L^2}\right)ds\right]\\
&\quad+\sum_k\beta_k^2\int_0^t\|h^k\|_{L^2}^2\,ds+\,2\mathbb{E}\left[\sum_k\int_0^t(h^k,u(s-))_{L^2}dZ_s^k\right]\\
&\quad+\sum_k\mathbb{E}\left[\sum_{0<s\le t}\|h^k\Delta Z_s^k\|_{L^2}^2\right]. \qquad (2.14)
\end{aligned}$$

It is easy to show

$$\begin{aligned}
\mathbb{E}\left[\beta_k^2\int_0^t\|h^k\|_{L^2}^2\,ds\right] &= \mathbb{E}\left[\beta_k^2\int_0^t\ \sigma^{ik}u_{x^i}+\nu^ku+g^k\|_{L^2}^2dt\right]\\
&\le 2\mathbb{E}\left[\int_0^t(\alpha_1^{ij}u_{x^i},u_{x^j})_{L^2}ds\right]+\varepsilon\|Du\|_{\mathbb{L}(t)}^2+c(\varepsilon)\|u\|_{\mathbb{L}(t)}^2+c(\varepsilon)\|g\|_{\mathbb{L}(t,\ell^2)}^2,
\end{aligned}$$

where $\alpha_1^{ij}=\frac{1}{2}\sum_k\beta_k^2\sigma^{ik}\sigma^{jk}$. Also,

$$\begin{aligned}
&\sum_k\mathbb{E}\left[\sum_{0<s\le t}\|h^k\Delta Z_s^k\|_{L^2}^2\right]\\
&= \sum_k\widehat{c}_k^2\mathbb{E}\left[\int_0^t\|\sigma^{ik}u_{x^i}+\mu^ku+g^k\|_{L^2}^2ds\right]\\
&\le 2\mathbb{E}\left[\int_0^t(\alpha_2^{ij}u_{x^i},\,u_{x^j})ds\right]+\varepsilon\|Du\|_{\mathbb{L}(t)}^2+c(\varepsilon)\|u\|_{\mathbb{L}(t)}^2+c(\varepsilon)\|g\|_{\mathbb{L}(t,\ell_2)}^2,
\end{aligned}$$

where $\alpha_2^{ij}=\frac{1}{2}\sum_k\widehat{c}_k^2\sigma^{ik}\sigma^{jk}$. Similarly,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t\left((\bar{b}^i,u_{x^i})+\sum_{i=1}^{d}(+f_i,u_{x^i})+(b^iu_{x^i}+cu+f_0,u)\right)ds\right]\\
&\le \varepsilon\|Du\|_{\mathbb{L}(t)}^2+c(\varepsilon)\|u\|_{\mathbb{L}(t)}^2+c(\varepsilon)\sum_{i=0}^{d}\|f_i\|_{\mathbb{L}(t)}^2.
\end{aligned}$$

Thus we have from (2.14) that for each $t \le T$,

$$\begin{aligned}
&\mathbb{E}\left[\|u(t)\|_{L^2}^2\right] + 2\mathbb{E}\left[\sum_{i,j=1}^{d}\int_0^t ((a^{ij}-\alpha^{ij})u_{x^i},\, u_{x^j})\right] ds \\
\le\ & \mathbb{E}\left[\|u_0\|_{L^2}^2\right] + \varepsilon\|Du\|_{\mathbb{L}(t)}^2 + c(\varepsilon)\int_0^t \mathbb{E}\left[\|u(s)\|_{L^2}^2\right] ds + c(\varepsilon)\sum_{i=0}^{d}\|f_i\|_{\mathbb{L}(t)}^2 + c(\varepsilon)\|g\|_{\mathbb{L}(t,\ell^2)}^2.
\end{aligned}$$

On the other hand, we know from condition (2.3) that

$$\sum_{i,j=1}^{d}((a^{ij}-\alpha^{ij})u_{x^i},\, u_{x^j}) \ge \delta\|Du\|_{L^2}^2.$$

The above two displays together with Grownwell's inequalty yield

$$\|u\|_{\mathbb{H}^1(T)} \le ce^{cT}\left(\|u_0\|_{U_2} + \|f\|_{\mathbb{H}^{-1}(T)} + \|g\|_{\mathbb{L}(T,\ell^2)}\right),$$

where $c = d(\delta, K)$. The lemma is proved. □

Remark 2.9 *The proof of Lemma 2.8 shows that if $\bar{b}^i = b^i = c = \nu^k = 0$, then*

$$\|u_x\|_{\mathbb{L}(T)} \le c\left(\|f\|_{\mathbb{H}^{-1}(T)} + \|g\|_{\mathbb{L}(T,\ell^2)} + \|u_0\|_{U_2}\right)$$

where c is **independent** *of T.*

For $\lambda \in [0,1]$, denote

$$a_\lambda^{ij} = \lambda a^{ij} + (1-\lambda)\delta^{ij}, \quad \sigma_\lambda^{ik} = \lambda\sigma^{ik},$$

$$\bar{b}_\lambda^i = \lambda\bar{b}^i, \quad b_\lambda^i = \lambda b^i, \quad c_\lambda = \lambda c, \quad \mu_\lambda^k = \lambda\mu^k.$$

$$L_\lambda u := \lambda Lu + (1-\lambda)\Delta u = \frac{\partial}{\partial x_i}(a_\lambda^{ij}u_{x^j} + \bar{b}_\lambda^i) + b_\lambda^i u_{x^i} + c_\lambda u,$$

$$\Lambda_\lambda^k u := \lambda\Lambda^k u := \sigma_\lambda^{ik}u_{x^i} + \mu_\lambda^k u \qquad \text{for } k \ge 1.$$

Note that

$$L_{\lambda_1}u - L_{\lambda_2}u = (\lambda_1-\lambda_2)(L-\Delta)u, \qquad \Lambda_{\lambda_1}u - \Lambda_{\lambda_2}u = (\lambda_1-\lambda_2)\Lambda u,$$

where $\Lambda_\lambda u := (\Lambda_\lambda^1 u, \Lambda_\lambda^2 u, \cdots)$, $\Lambda u := (\Lambda^1 u, \Lambda^2 u, \cdots)$, and

$$\|L_{\lambda_1}u - L_{\lambda_2}u\|_{H^{-1}} + \|\Lambda_{\lambda_1}u - \Lambda_{\lambda_2}u\|_{L^2(\ell^2)} \le c|\lambda_1-\lambda_2|\|u\|_{H^1}. \tag{2.15}$$

Remark 2.10 It is trivial to check that a priori estimate (2.13) holds with the same constant C if $u \in \mathcal{H}^1(T)$ is a solution of the equation obtained by replacing the coefficients $a^{ij}, \bar{b}^i, \cdots, \mu^k$ in (2.4) by $a_\lambda^{ij}, \bar{b}_\lambda^i, \cdots, \mu_\lambda^k$, respectively, for every $\lambda \in [0,1]$.

Here is the main result of this section.

Theorem 2.11 *Suppose* **Assumptions 2.1** *and* **2.7** *hold. Then for every* $f \in \mathbb{H}^{-1}(T)$, $g \in \mathbb{L}(T, \ell^2)$ *and* $u_0 \in U_2$, *equation (2.4) has a unique solution* $u \in \mathcal{H}^1(T)$ *with* $u(0) = u_0$, *and*

$$\|u\|_{\mathcal{H}^1(T)} \leq ce^{cT}(\|f\|_{\mathbb{H}^{-1}(T)} + \|g\|_{\mathbb{L}(T,\ell^2)} + \|u_0\|_{U_2}), \tag{2.16}$$

where $c = c(\delta, K)$.

Proof. In view of the a priori estimate in Lemma 2.8, it suffices to show that there is a solution to (2.4). First, we show that for any given $f \in \mathbb{H}^{-1}(T), g \in \mathbb{L}(T, \ell^2)$ and $u_0 \in U_2$, the equation

$$du = (\Delta u + f)dt + g^k dZ_t^k, \quad u(0) = u_0 \tag{2.17}$$

has a solution $u \in \mathcal{H}^1(T)$. Due to a priori estimate (2.13), Remark 2.10 and standard approximation argument, we may assume that f, u_0 are infinitely differentiable in x with compact supports. Also by the same reasoning (also see Theorem 3.10 in [7]), we may assume that $g^k = 0$ for all $k \geq N$ for some $N \geq 1$, and g^k is of the type

$$g^k(t) = \sum_{i=1}^m I_{(\tau_{i-1}, \tau_i]}(t)\varphi_i(x),$$

where τ_i are bounded stopping times and $\varphi_i \in C_c^\infty(\mathbb{R}^d)$. Define

$$v(t) = \sum_{k=1}^N \int_0^t g^k(s)dZ_s^k.$$

Then it is easy to see that $v \in \mathcal{H}^1(T)$. Note that u satisfies (2.17) if and only if $\bar{u} = u - v$ satisfies

$$d\bar{u} = (\Delta\bar{u} + \Delta v + f)dt \qquad \text{with} \quad \bar{u}(0) = u_0.$$

Since this equation has a solution in $\mathcal{H}^1(T)$ (see Theorem 5.1 in [7]), we conclude that equation (2.17) has a solution u in $\mathcal{H}^1(T)$.

Let $J \subset [0, 1]$ denote the set of λ, so that for any f, g, u_0, the equation

$$du = (L_\lambda u + f)dt + (\Lambda_\lambda^k u + g^k)dZ_t^k, \quad u(0) = u_0 \tag{2.18}$$

has a solution $u \in \mathcal{H}^1(T)$. Then as proved above, $0 \in J$. Now assume $\lambda_0 \in J$, and note that u is a solution of equation (2.18) if and only if

$$du = (L_{\lambda_0}u + (L_\lambda u - L_{\lambda_0}u + f))dt + (\Lambda_{\lambda_0}u + (\Lambda_\lambda^k u - \Lambda_{\lambda_0}^k u + g^k))dZ_t^k. \tag{2.19}$$

Note that $D : H^n \to H^{n-1}$ is a bounded operator. Thus for any $u \in \mathcal{H}^1(T)$, $k \geq 1$ and $\lambda \in [0, 1]$, we have

$$L_\lambda u \in \mathbb{H}^{-1}(T) \qquad \text{and} \qquad \Lambda_\lambda u \in \mathbb{L}(T, \ell^2).$$

Recall $\lambda_0 \in J$. Denote $u^0 = u_0$ and for $n = 1, 2, \cdots$ we define $u^{n+1} \in \mathcal{H}^1(T)$ as the solution of the equation

$$du^{n+1} = (L_{\lambda_0} u^{n+1} + f_n)dt + (\Lambda_{\lambda_0} u^{n+1} + g_n^k)dZ_t^k, \quad u^{n+1}(0) = u_0$$

where

$$f_n := L_\lambda u^n - L_{\lambda_0} u^n + f \quad \text{and} \quad g_n^k := \Lambda_\lambda^k u^n - \Lambda_{\lambda_0}^k u^n + g^k.$$

By Remark 2.10 and inequality (2.15), we have

$$\begin{aligned} \|u^{n+1} - u^n\|_{\mathcal{H}^1(T)} &\leq c\|(L_\lambda - L_{\lambda_0})(u^n - u^{n-1})\|_{\mathbb{H}^{-1}(T)} + c\|(\Lambda_\lambda - \Lambda_{\lambda_0})(u^n - u^{n-1})\|_{\mathbb{L}(T)} \\ &\leq c\|\lambda - \lambda_0\|\|u^n - u^{n-1}\|_{\mathbb{H}^1(T)}. \end{aligned}$$

Let $\varepsilon_0 = c/2$. Then for $\lambda \in (\lambda_0 - \varepsilon_0,\, \lambda + \varepsilon_0)$, $\|u^{n+1} - u^n\|_{\mathcal{H}^1(T)} \leq \frac{1}{2}|\lambda - \lambda_0|\|u^n - u^{n-1}\|_{\mathbb{H}^1(T)}$ for every $n \geq 1$ and so u^n converges to some u in $\mathcal{H}^1(T)$. It follows that u solves equation (2.19). This proves that $(\lambda_0 - \varepsilon_0,\, \lambda_0 + \varepsilon_0) \cap [0, 1] \subset J$. Consequently we conclude $J = [0, 1]$. □

The following remark plays the key role when we weaken condition (2.2) later in the next section.

Remark 2.12 *Let $\tau \leq T$ be a stopping time. We use $1_{[\![0,\tau]\!]}$ to denote the random process $t \mapsto 1_{[0,\tau]}(t)$. For an H^1-valued $\mathcal{P}^{dP \times dt}$-measurable process u, write $u \in \mathbb{H}^1(\tau)$ if*

$$\|u\|^2_{\mathbb{H}^1(\tau)} := \mathbb{E}\left[\int_0^\tau \|u\|^2_{H^1} ds\right] < \infty.$$

Similarly define $\mathbb{L}(\tau, \ell_2)$ and $\mathcal{H}^1(\tau)$. Then Theorem 2.11 holds with the deterministic time T replaced by the stopping time τ. Indeed, the existence of solution $u \in \mathcal{H}^1(\tau)$ and the estimate (2.16) are easily obtained by applying Theorem 2.11 with $\bar{f} = f1_{[\![0,\tau]\!]}$ and $\bar{g} = g1_{[\![0,\tau]\!]}$ in place of f and g, respectively. Now let $u \in \mathcal{H}^1(\tau)$ be a solution. According to Theorem 2.11 we can define $v \in \mathcal{H}^1(T)$ as the solution of

$$dv = (\Delta v + (\mathbb{D}u - \Delta u)1_{[\![0,\tau]\!]})dt + 1_{[\![0,\tau]\!]}\mathbb{S}^k u\, dZ_t^k, \quad v(0) = u(0). \tag{2.20}$$

Then for $t \leq [0, \tau)$, $d(u - v) = \Delta(u - v)dt$ and therefore we conclude that $u(t) = v(t)$ for all $t \leq \tau$, a.s.. Thus, equation (2.20) becomes

$$\begin{aligned} dv &= \left(\sum_{i=1}^d \frac{\partial}{\partial x^i}\left(\sum_{j=1}^d a_\tau^{ij} v_{x^j} + \bar{b}_\tau^i v\right) + b_\tau^i v_{x^i} + c_\tau v + f1_{[\![0,\tau]\!]}\right) dt \\ &\quad + \sum_{k\geq 1}\left(\sum_{i=1}^d \sigma_\tau^{ik} v_{x^i} + \mu_\tau^k v + g^k 1_{[\![0,\tau]\!]}\right) dZ_t^k, \end{aligned} \tag{2.21}$$

where

$$a_\tau^{ij} = a^{ij}1_{[\![0,\tau]\!]} + \delta^{ij}1_{]\!]\tau,\infty[\![}, \quad \bar{b}_\tau^i = \bar{b}^i 1_{[\![0,\tau]\!]}, \quad b_\tau^i = b^i 1_{[\![0,\tau]\!]}, \quad \cdots, \mu_\tau^k = \mu^k 1_{[\![0,\tau]\!]}.$$

The uniqueness result of equation (2.4) in the space $\mathcal{H}^1(\tau)$ follows from the uniqueness result of equation (2.21) in $\mathcal{H}^1(T)$.

3 Some extensions

In this section we give two extensions of Theorem 2.11. First, we consider the nonlinear equation

$$
\begin{aligned}
du &= \left(\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} a^{ij} u_{x^j} + \bar{b}^i u \right) + \sum_{i=1}^{d} b^i u_{x^i} + cu + f(u) \right) dt \\
&\quad + \sum_{k\geq 1} \left(\sum_{i=1}^{d} \sigma^{ik} u_{x^i} + \mu^k u + g^k(u) \right) dZ_t^k, \qquad (3.1)
\end{aligned}
$$

where $f(u) = f(\omega, u, t, x)$ and $g^k(u) = g^k(\omega, u, t, x)$.

Assumption 3.1 (i) For any $u \in \mathbb{H}^1(T)$,

$$f(u) \in \mathbb{H}^{-1}(T) \quad \text{and} \quad g(u) := (g^1(u), g^2(u), \cdots) \in \mathbb{L}(T, \ell^2).$$

(ii) For every $\varepsilon > 0$, there exists a constant $K_1 = K_1(\varepsilon)$ so that for every $t \in (0, T]$ and $u, v \in \mathbb{H}^1(t)$,

$$\|f(u) - f(v)\|^2_{\mathbb{H}^{-1}(t)} + \|g(u) - g(v)\|^2_{\mathbb{L}(t,\ell^2)} \leq \varepsilon \|u - v\|^2_{\mathbb{H}^1(t)} + K_1 \|u - v\|^2_{\mathbb{L}(t)}. \qquad (3.2)$$

Theorem 3.2 *Suppose* **Assumptions 2.1**, **2.7** *and* **3.1** *hold. Then for any $u_0 \in U_2$, equation (3.1) with initial data $u_0 \in U_2$ has a unique solution $u \in \mathcal{H}^1(T)$, and*

$$\|u\|_{\mathcal{H}^1(T)} \leq c(\|f(0)\|_{\mathbb{H}^{-1}(T)} + \|g(0)\|_{\mathbb{L}(T,\ell^2)} + \|u_0\|_{U_2}) \qquad (3.3)$$

where $f(0) = f(\omega, 0, t, x), g(0) = g(\omega, 0, t, x)$ and $c = c(\delta, K, T) > 0$.

Proof. We will use fixed point theorem to show the existence and uniqueness of the solution to (3.1). Estimate (3.3) follows from (2.13), condition (3.2) and the Grownwall's inequality. Let $\mathcal{R}(f, g) \in \mathcal{H}^1(T)$ denote the solution of (2.4) with initial data u_0. Then by Theorem 2.11,

$$\mathcal{R}u := \mathcal{R}(f(u), g(u)) \qquad \text{for } u \in \mathcal{H}^1(T)$$

is well defined and $\mathcal{R}$ is a map from $\mathcal{H}^1(T)$ to $\mathcal{H}^1(T)$. Define $u^0 = \mathcal{R}(f(0), g(0))$ and $u^{n+1} = \mathcal{R}(f(u^n), g(u^n))$. Then by Theorem 2.11 and assumption (3.2), for any $t \leq T$,

$$
\begin{aligned}
\|\mathcal{R}u - \mathcal{R}v\|^2_{\mathcal{H}^1(t)} &\leq c\varepsilon \|u - v\|^2_{\mathcal{H}^1(t)} + cK_1 \|u - v\|^2_{\mathbb{L}(t)} \\
&\leq c\varepsilon \|u - v\|^2_{\mathcal{H}^1(t)} + cK_1 \int_0^t \|u - v\|^2_{\mathcal{H}^1(s)} ds
\end{aligned}
$$

where the last inequality is from (2.6). Taking $\varepsilon = 1/(2c)$ and then letting $t_0 > 0$ be small enough so that $cK_1 t_0 < 1/4$, we have

$$\|\mathcal{R}u - \mathcal{R}v\|^2_{\mathcal{H}^1(t_0)} \leq \frac{1}{2} \|u - v\|^2_{\mathcal{H}^1(t_0)} \qquad (3.4)$$

This contraction implies that u^n converges to some u in $\mathcal{H}^1(t_0)$ and u is a solution to (3.1) on $[0,t_0]$ with initial value u_0. The inequality (3.4) further implies the uniqueness of solution to (3.1) on $[0,t_0]$ initial value u_0. Iterating this procedure at most $[T/t_0]+1$ many time intervals of size no larger than t_0 and using estimate (2.6), we get the desired results on time interval $[0,T]$. For more details, we refer the reader to the proof of Theorem 6.4 in [7]. □

Example 3.3 *Let's consider an equation with fractional Laplacian. For simplicity assume $g^k(u)=0$ for $k\geq 2$. Take $f(u)=(-\Delta)^{\alpha/2}u$ and $g(u)=g^1(u)=(-\Delta)^{\beta/2}u$ where $\alpha<2$ and $\beta<1$, then obviously for any $\varepsilon>0$,*

$$\begin{aligned}\|f(u)-f(v)\|^2_{\mathbb{H}^{-1}(t)}+\|g(u)-g(v)\|^2_{\mathbb{L}(t)} &\leq c\|u-v\|^2_{\mathbb{H}^{-1+\alpha}(t)}+c\|u-v\|^2_{\mathbb{H}^{\beta}(t)}\\ &\leq \varepsilon\|u-v\|^2_{\mathbb{H}^1(t)}+K_1\|u-v\|^2_{\mathbb{L}(t)},\end{aligned}$$

where for the second inequality we use the following fact: if $\gamma=\kappa\gamma_1+(1-\kappa)\gamma_0$ and $\kappa\in[0,1]$ then $\|u\|_{H^\gamma}\leq N\|u\|^{\kappa}_{H^{\gamma_1}}\|u\|^{1-\kappa}_{H^{\gamma_0}}$. Thus the existence and uniqueness of equation (3.3) in $\mathcal{H}^1(T)$ is guaranteed by Theorem 3.2.

For a stopping time $\tau\in(0,T]$ write $u\in\mathbb{H}^1_{\rm loc}(\tau)$ if there exists a sequence of stopping times $\tau_n\uparrow\infty$ so that $u\in\mathbb{H}^1(\tau\wedge\tau_n)$ for each n.

The following is a weakened version of **Assumption 2.1**.

Assumption 3.4 *There exists an integer $N_0\geq 1$ so that*

(i) *$\widehat{c}_k<\infty$ for all integer $k>N_0$;*

(ii) *for some $\delta>0$,*

$$(a^{ij}-\alpha^{ij}_{N_0})_{d\times d}>\delta I_{d\times d}, \tag{3.5}$$

where $\alpha^{ij}_{N_0}:=\frac{1}{2}\sum_{k=N_0+1}^{\infty}(\widehat{c}^2_k+\beta^2_k)\sigma^{ik}\sigma^{jk}$.

Here is our second extension.

Theorem 3.5 *Let Assumption 3.4 hold and $\sigma^i_k=0$ for $k\leq N_0$. Then for any $u_0\in U_2$, $f\in\mathbb{H}^{-1}(T)$ and process $g=(g^1,g^2,\cdots)$ having entries in $\mathbb{L}_2(T)$ so that $\sum_{N_0+1}^{\infty}\widehat{c}^2_k\|g^k\|^2_{\mathbb{L}(T)}<\infty$, there exists unique $u\in\mathbb{H}^1_{\rm loc}(T)$ such that*

(i) *$u(t)$ is right continuous with left limits in L^2 a.s.,*

(ii) *for any $\phi\in C^\infty_c(\mathbb{R}^d)$, the equality*

$$\begin{aligned}(u(t),\phi) &= (u_0,\phi)+\int_0^t\big((-a^{ij}u_{x^j}-\bar{b}^iu,\,\phi_{x^i})+(b^iu_{x^i}+cu+f,\,\phi)\big)\,ds\\ &\quad+\int_0^t\Big((\sigma^{ik}u_{x^i},\phi)+(\mu^k,\phi)+(g^k,\phi)\Big)\,dZ^k_s\end{aligned} \tag{3.6}$$

holds for all $t<T$ a.s..

Proof. **Step 1**. First assume that Assumption 2.1 holds, that is, $\widehat{c}_k < \infty$ for each k. Let $\tau \leq T$ be a stopping time. We show that the pathwise solution is unique in $\mathbb{H}^1_{\text{loc}}(\tau)$. Let $u \in \mathbb{H}^1_{\text{loc}}(\tau)$ be a path-wise solution, that is, u satisfies the conditions (i) and (ii) in the theorem for $t < \tau$. Define $\tau_n = \tau \wedge \inf\{t : \int_0^t \|u\|^2_{H^1} ds > n\}$. Then $u \in \mathbb{H}^1(\tau_n)$ and $\tau_n \uparrow \tau$ since $\int_0^t \|u\|^2_{H^1} ds < \infty$ for all $t < \tau$, a.s. By Remark 2.12,

$$\|u\|_{\mathbb{H}^1(\tau_n)} \leq c(T, d, K)(\|f\|_{\mathbb{H}^{-1}(\tau_n)} + \|g\|_{\mathbb{L}(\tau_n, \ell_2)} + \|u(0)\|_{U_2}).$$

By letting $n \to \infty$ we find that $u \in \mathcal{H}^1(\tau)$, and the uniqueness of the pathwise solution under Assumption 2.1 follows from Remark 2.12. Note that the existence of pathwise solution under Assumption 2.1 in $\mathbb{H}^1(\tau)$ also follows from Theorem 2.11.

Step 2. For the general case, note that for each $n > 0$

$$\widehat{c}_{k,n} := \left(\int_{\{z \in \mathbb{R} : |z| \leq n\}} |z|^2 \nu_k(dz) \right)^{1/2} \qquad \text{for } k \leq N_0.$$

Consider the Lévy processes $(Z^1_n, \cdots, Z^{N_0}_n, Z^{N_0+1}, \cdots)$ in place of $(Z^1, Z^2 \cdots)$, where $Z^k_n (k \leq N_0)$ is a Lévy process obtained from Z^k by removing all the jumps that has absolute size strictly large than n. Note that condition (2.3) is valid with $\widehat{c}_k$ replaced by $\widehat{c}_{k,n}$ since σ^{ik} are assumed to be zero for $k \leq N_0$.

By Step 1, there is a unique pathwise solution $v_n \in \mathcal{H}^1(T)$ with Z^k_n in place of Z^k for $k = 1, 2, \cdots, N_0$. Let T_n be the first time that one of the Lévy processes $\{Z^k, 1 \leq k \leq N_0\}$ has a jump of (absolute) size in (n, ∞). Define $u(t) = v_n(t)$ for $t < T_n \wedge T$. Note that for $n < m$, by Step 1, we have $v_n(t) = v_m(t)$ for $t < T_n$. This is because, for $t < T_n$, both v_n and v_m satisfies (3.6) with each term inside the stochastic integral multiplied by $1_{s<T_n}$ (and with Z^k_n, $k \leq N_0$, in place of Z^k). Thus u is well defined. By letting $n \to \infty$, one constructs unique pathwise solution u in $\mathbb{H}^1_{loc}(T)$. The theorem is proved.

□

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